A New Approach To Option Pricing for Discrete Hedging and Non-Gaussian Processes

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Introduction

Motivation

Aim for an option pricing method

- for (any) real underlying probability distribution
- with an algorithm independent
  on the process
⇒ fair value, volatility surface
⇒ hedging strategy

⇒ Bouchaud-Sornette method\(^1\) has these features
⇒ works in discrete time, i.e. discrete hedging

⇒ Risk cannot be eliminated completely
⇒ Option price is not unique

Outline

1. Pricing methods
   • Assumptions
   • Bouchaud-Sornette, Minimal Price Approach

2. Comparison of the methods
   • Dependency on the hedging frequency
   • Comparison of the hedging strategies
   • Dependency on the price of risk
   • Volatility matrix (smile, term structure)
   • Dependency on the drift of the underlying
Introduction

Assumptions

• Trading in discrete time (equidistant time steps)
• “Arbitrary” i.i.d. process. Probability density function for the log-return.
• No bid/ask spread.
• Short selling is allowed.
• Interest rate is constant. Interest curve is flat.
• No dividends
• No transaction cost
• Option style European, payoff is path independent.
Pricing Methods

for Black-Scholes, Bouchaud-Sornette and the Minimal Price Approach

Hedging Portfolio
Value (Hedging Strategy)

Price Equation
Option Price (Hedging-Strategy)

Hedging Strategy
Optimisation of Strategy

Option Price

Economical Argument

Market Price

Methods are different here
Methods are different here

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Hedging Portfolio = Underlying – Option + Cash

at time $T$:

Option premium $V_0 B^{-m}$

Option (short) $- V(T)$

Hedging (equity + cash) $\sum_{l=0}^{m-1} \phi_l \cdot \Delta \tilde{S}_l$

Portfolio value:

$\Pi(T) = V_0 B^{-m} - V(T) + \sum_{l=0}^{m-1} \phi_l \cdot \Delta \tilde{S}_l$

$\Delta \tilde{S}_l = (S_{l+1} \cdot B - S_l) \cdot B^{l-m}$
Pricing Methods

for Black-Scholes, Bouchaud-Sornette and the Minimal Price Approach

- Hedging Portfolio Value (Hedging Strategy)
- Price Equation
  - Option Price (Hedging-Strategy)
- Hedging Strategy
  - Optimisation of Strategy
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Methods are different here
Price Equation

For any remaining risk, a premium is to charge.

\[
\langle \Pi(T) \rangle_0 - \langle \Pi(t_0) \cdot B^{m} \rangle_0 = \lambda \cdot R(T)
\]

Price of Risk  Risk  \[ R(T) = \sqrt{\langle \Pi^2(T) \rangle_0 - \langle \Pi(T) \rangle_0^2} \]

Substitution of Pi(T):

\[
V_0 \cdot B^{-m} - \langle V(T) \rangle_0 + \sum_{l=0}^{m-1} \langle \phi_l \cdot \Delta \tilde{S}_l \rangle_0 = \lambda \cdot R(T)
\]

The risk is independent on \( V_0 \).

\[
V_0 \cdot B^{-m} = \lambda \cdot R(T) + \langle V(T) \rangle_0 - \sum_{l=0}^{m-1} \langle \phi_l \cdot \Delta \tilde{S}_l \rangle_0
\]
Pricing Methods
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Derivation of the Hedging Strategy

The hedging strategy $\phi(S,t)$ is determined by a "global strategy"

Black-Scholes: elimination of risk
Bouchaud-Sornette: risk is minimal
Minimal Price Approach option price is minimal

e.g.

Bouchaud-Sornette:
\[ \frac{\partial R}{\partial \phi} = 0 \]

Minimal Price Approach
\[ \frac{\partial V_0}{\partial \phi} = 0 \]
Bouchaud-Sornette Method

Minimal risk:

\[
\frac{\partial R}{\partial \phi} = 0
\]

\[
\phi(S_k, t_k) = \frac{1}{\langle \Delta \tilde{S}_k^2 \rangle_k} \left\{ \left\langle V(T) \cdot \Delta \tilde{S}_k \right\rangle_k + \langle \Delta \tilde{S}_k \rangle_k \cdot \left[ \sum_{l=0}^{m-1} \left\langle \phi_l \cdot \Delta \tilde{S}_l \right\rangle_0 - \left\langle V(T) \right\rangle_0 \right] - \sum_{l=0}^{m-1} \sum_{l \neq k}^{m-1} \left\langle \phi_l \cdot \Delta \tilde{S}_l \cdot \Delta \tilde{S}_k \right\rangle_k \right\}
\]

can be solved by an iteration procedure.

Initial value:

\[
\phi(S_k, t_k) = \frac{\left\langle V(T) \cdot \Delta \tilde{S}_k \right\rangle_0}{\langle \Delta \tilde{S}_k^2 \rangle_k}
\]
Minimal Price Approach

Minimal price:
\[
\frac{\partial V_0}{\partial \phi} = 0
\]

\[
\phi(S_k, t_k) = \frac{1}{\langle \Delta \tilde{S}_k^2 \rangle_k} \left\{ \langle V(T) \cdot \Delta \tilde{S}_k \rangle_k + \langle \Delta \tilde{S}_k \rangle_k \cdot \frac{1}{\lambda} \cdot R(T) + \langle \Delta \tilde{S}_k \rangle_k \cdot \left[ \sum_{l=0}^{m-1} \langle \phi_l \cdot \Delta \tilde{S}_l \rangle_0 - \langle V(T) \rangle_0 \right] - \sum_{l=0}^{m-1} \langle \phi_l \cdot \Delta \tilde{S}_l \cdot \Delta \tilde{S}_k \rangle_k \right\}
\]

- Extra term compared to Bouchaud-Sornette
- \( \lambda \to \infty \): Bouchaud-Sornette = Minimal Price Method
- Continuous time + log-normal distribution: both methods converge towards the Black-Scholes result
Dependency on the Price of Risk

hedging strategy:
- Black-Scholes without risk premium
- Black-Scholes
- Bouchaud-Sornette
- Minimal Price Approach

Plain Vanilla Call
at the money, fat tails

\[
\text{premium} = \frac{\text{equity drift - interest rate}}{\text{equity volatility}} \times \sqrt{\text{option term}} = \frac{0.1 - 0.05}{0.2} \times \sqrt{0.2} = 0.11
\]

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Economical Argument

Black-Scholes:
no-arbitrage argument

Bouchaud-Sornette:
missing

Minimal Price Approach:
market price is given by the best price

Model dependency:
price of risk
risk measure

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Case Study

Comparison of Methods

Process for the Underlying

<table>
<thead>
<tr>
<th>Process</th>
<th>log-Normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameters</td>
<td>$\sigma = 0.2, \mu = 0.1$</td>
</tr>
</tbody>
</table>

Options

<table>
<thead>
<tr>
<th>Option Type</th>
<th>European Plain Vanilla Call/Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>European Binary Call/Put</td>
<td></td>
</tr>
</tbody>
</table>

Term

| Term     | 0.2 years |

Other Parameters

<table>
<thead>
<tr>
<th>Interest Rate</th>
<th>5 %</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price of Risk</td>
<td>$= 1.1 \cdot \text{Price of Risk (Underlying)} \approx 0.12$</td>
</tr>
</tbody>
</table>
Case Study

Option Premium at the money
log-Normal distribution

„Fat Tail“ distribution
Case Study

Option Premium
out of the money

- Black-Scholes hedging ≈ Bouchaud-Sornette hedging
- Premium is up to 10% cheaper for the Minimal Price Approach
- Put / Call, Plain Vanilla / Binary: similar results
- Prices are higher than Black-Scholes price
Case Study

Comparison of Hedging Strategies

Risk (MPA) ≈ 2 · Risk (B.-S.) for this example

Bouchaud-Sornette:
\[ <\text{Residual Profit (S)} > \approx 0 \]
no correlation with S(T)

Portfolio theory:
Hedging-Portfolio + Underlying improves return/risk ratio

Minimal Price Approach:
positive correlation with S(T)

⇒ increased stock position
Case Study

Comparison of Hedging Strategies

Minimal Price Approach:
increased stock position

Explanation by portfolio theory
⇒ Is there a drift dependency?
Case Study

Dependency on the Drift

Bouchaud-Sornette / Black-Scholes

No drift dependency

Minimal Price Approach:

Drift dependency $\approx$
extra stocks hold $\cdot \Delta \text{Drift} \cdot T \cdot S$
$\approx 0.2 \cdot 0.02 \cdot 0.2 \cdot 50 \approx 0.04$  (for this example)

$\Rightarrow$  additional risk
risk premium $\approx 0.12 \cdot \Delta$ profit

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Case Study

Implied Volatility for Plain Vanilla Options

Minimal Price Approach

Implied volatility

0.235
0.23
0.225
0.22
0.215
0.21
0.205

0.1
0.2
0.3
60
50
40

Maturity
Strike
Case Study

Volatility Smile for Plain Vanilla Options

Smile is due to:
• Fat Tails
• Risk premium

![Graphs showing volatility smile for plain vanilla options with explanations for fat tails and risk premium.]
Case Study

Volatility Smile for Plain Vanilla Options

Asymmetric probability distribution ⇒ asymmetric smile

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Term Structure for Plain Vanilla Options

At the Money and „Fat Tails“:
impl. volatility increases with maturity,

\[ V(S = E) \approx \frac{1}{2} \cdot \langle |S - E| \rangle_0 \]

and \( \langle |S - E| \rangle_0 \) increasing with \( T \).

S: underlying price
E: strike

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Scaling with Maturity
Plain Vanilla Options

Log-Normal distribution:
Width of the smiles scales with $\sqrt{dt}$

Smile is symmetric as function of $\log(\text{Strike}/\text{Spot})$

$$Vola(E, T) \approx Vola\left(\frac{1}{\sqrt{T}} \cdot \ln\frac{E}{\text{Spot}}\right)$$

E: strike
Case Study

Scaling with Maturity
Plain Vanilla Options

„Fat Tail“ distribution:
\[ \sqrt{dt} \text{ scaling only for long-term options} \]
\[ \Rightarrow \text{process distribution converges towards a Gaussian} \]

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Case Study

Implied Volatilities for Binary Options

Minimal Price Approach

[Graph showing implied volatilities for binary options with strike and maturity on the axes and implied volatility on the Y-axis.]
Implied Volatilities for Binary Options

1. $V \approx 1$, due to the risk premium the price exceeds any possible Black-Scholes price.

2. Black-Scholes price heavily depends on the volatility. Risk premium makes impl. volatility smaller.

3. Due to the risk premium the price exceeds any possible Black-Scholes price.
Conclusions

Pricing Methods
- Methods for discrete hedging converges towards Black-Scholes results for a log-normal distribution in the continuous time limit.
- Bouchaud-Sornette hedging \( \approx \) Black-Scholes hedging
- Minimal Price Approach is most competitive on the market

Effect of Fat Tails
- „Fat Tails“ give rise to volatility smiles und a slope in the term structure
- „Fat Tails“ spoil the \( \frac{1}{\sqrt{T}} \) - scaling of the smile

Risk Premium
- Risk premium cannot be neglected in the price.
- Risk premium gives rise to a volatility smile.
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